Bondi — Sachs Formalism

1. Introduction

   - prelims
   - how GR is different from EM?

2. Different approaches

3. Gravity in Bondi gauge

   Bondi News

   Bondi mass

outline lecture 1

Bondi Sachs Formalism

Introduction

(a) Prelims

1. In General Relativity we are often interested in studying properties of isolated systems.

2. In practice no physical system is truly isolated.

3. Say, you want to study properties of a star, then you should be able to ignore cosmological curvature, distant galaxies and study the problem as if star is located in flat space.

4. Flat space — vanishing gravitational field.

5. Asymptotically flat spacetimes are isolated systems in GR. GR = General Relativity.

6. Depending on the context and situation you might be interested in other set ups.

   ▷ Asymptotically anti - de - sitter

- ▷ Asymptotically de Sitter
- ▷ Asymptotically cosmological spacetimes

7 We may be interested, but the key difficulties are in defining these notions.

(b) How GR is different from electrodynamics?

8 In electrodynamics there is flat space and we want to study isolated charge / current distributions.

9 We can specify the fall off rates for the charge current J^μ and EM vector field A^μ

J^μ : compact support

A^μ is such that

static $\quad F_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{r^2}\right) \quad r \to \infty$ at fixed t

Radiative $\quad F_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{r}\right) \quad r \to \infty$ at fixed u

$u = t - r$

Big O notation :-

more common – what we will use

$$O\left(\frac{1}{r^2}\right) = \frac{c_1}{r^2} + \frac{c_2}{r^3} + \cdots$$

example

$$e^x = 1 + x + O(x^2)$$

absolute error in $e^x - 1 - x$ is at most $O|x^2|$ as $x \to 0$, i.e., some constant time $|x^2|$ for $x \to 0$.

small - o notation :-

equally common – not in our lectures

$$o(\rho^{-m})$$

$$\lim_{\rho \to \infty} \rho^m \, o(\rho^{-m}) = 0$$

This difference in important when working with log terms

$$\lim_{\rho \to \infty} \sim \frac{\ln \rho}{\rho^2}$$

$$\rho \, o\left(\frac{1}{\rho}\right) \sim \rho \frac{\ln \rho}{\rho^2} = \frac{\ln \rho}{\rho} \to 0$$

$$o\left(\frac{1}{\rho}\right) = \frac{\ln \rho}{\rho^2} + \frac{1}{\rho^2} + \cdots$$

typically if log - terms are present more careful notation is needed.

going back to electrodynamics,

(10) maxwell's equation tell us a great deal about the detailed structure and properties of EM field at large distances.

(11) multipole expansion of EM field with infinite set of multipole moments which are related to multipole moments of charge and current moments in a simple way.

both in stationary regime and dynamical regime.

$\boxed{\text{what about GR ?}}$

why are such things not taught in a GR course?

(12) There are good reasons for not teaching such things. There is no background structure. serious obstacle.

There is no natural radial coordinate to specify fall off rates.

what can we do ?

(2) Different approaches

(1) One way around this difficulty is to define Asymptotically Flat spacetimes if there exists any coordinate system x^0, x^1, x^2, x^3 s.t. metric components behave in an appropriate way at large distances

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right) \qquad r \to \infty.$$

either along spatial or null directions ....

(2) The above idea is correct and adequate in many respect, but is not manifestly coordinate independent. More over, it is difficult to specify precise fall-offs and how $r \to \infty$ limit is to be taken in a meaningful way.

(3) To address such concerns Penrose ~~gave~~ gave conformal infinity approach.

Much more popular in general relativity community

In this approach one attaches a boundary representing points at infinity to the physical spacetime in a suitable way ....

4 Such an approach is mathematically speaking cleaner but

(i) doing calculations is difficult

(ii) difficult to bring in richness of the coordinate based approach

Popular approaches	coordinates	geometrical
null	Bondi - Sachs	Penrose conformal infinity
spacelike or timelike	Beig - Schmidt	Geroch
	unified	unified
unified frameworks	coordinate relations	Ashtekar - Hansen

(5) I have been asked to teach Bondi-Sachs

- tied to coordinates
- some of the math issues are difficult to address
- historically this analysis provided the first convincing analysis of energy carried by gravitational radiation.

(6) It is very hands on. Much of the modern day research is done in the Bondi-Sachs formalism.

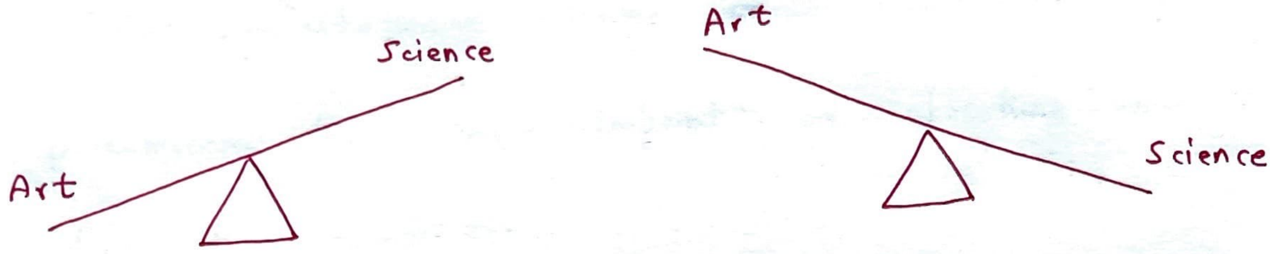
(7) Examples of the type of problems people are discussing

(i) Bondi-Sachs in de Sitter

(ii) coordinate vs. unified Ashtekar-Hansen formalism

- in the original papers the focus was only at spatial infinity
- more modern work has been to relate Beig-Schmidt to Bondi-Sachs.

very active field



what you think is interesting — art — how much freedom you want to give — what sort of physics questions it helps you address.

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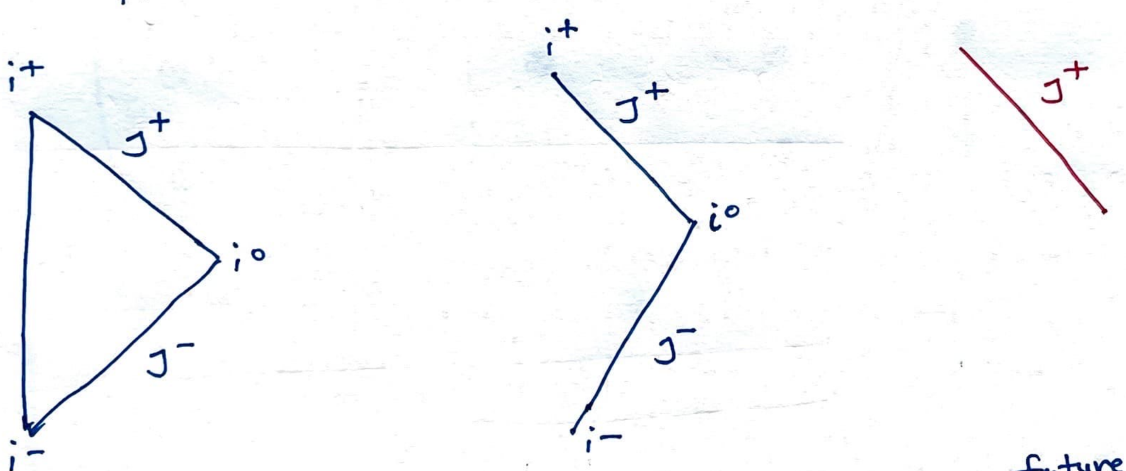
3 Gravity in Bondi gauge

1 we want to define asymptotically flat spacetimes which approach the notion of future null infinity



flat space

future, past, spacelike, timelike infinity

future null infinity

2 Physically — radiation zone — where gravitational waves and other null wave phenomenon leave their imprint on spacetime far from the sources.

$$x^\mu = (u, r, \theta, \phi)$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

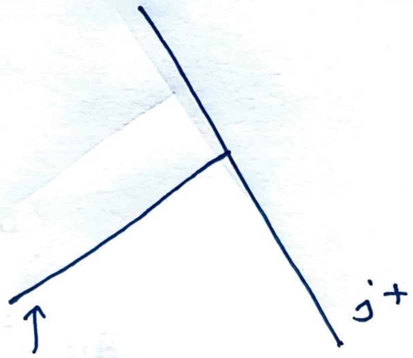
$$= g_{uu}\, du^2 + 2 g_{ur}\, du\, dr + 2 g_{uA}\, du\, dx^A$$

$$+ g_{AB}\, dx^A\, dx^B$$

$$\begin{array}{cc} \left(\begin{array}{cc|cc} \otimes & \otimes & \otimes & \otimes \\ \otimes & 0 & 0 & 0 \\ \hline \otimes & 0 & \otimes & \otimes \\ \otimes & 0 & \otimes & \otimes \end{array}\right) & \begin{array}{c} u \\ r \\ \theta \\ \phi \end{array} \\ \begin{array}{cccc} u & r & \theta & \phi \end{array} & \end{array}$$

$$\left.\begin{array}{l} g_{rr} = 0 \\ g_{rA} = 0 \\ \sqrt{\det g_{AB}} = r^2 f(\theta, \phi) \end{array}\right\} \text{Four gauge conditions}$$

coordinates are based on null hypersurfaces labeled by a constant u coordinate



u constant null hypersurface

$$g^{uu} = 0$$

The nornal $n^\mu = g^{\mu u} = g^{\mu\nu}\partial_\nu u$

angular coordinates are such that moving along the normal $x^A = (\theta, \phi)$ do not change.

$$n^\mu \partial_\mu \theta = 0 \qquad n^\mu \partial_\mu \phi = 0$$

$$n^\theta = 0 \Rightarrow g^{\theta u} = 0 \qquad g^{\phi u} = 0$$

Asympotics

so far it is just geometry.

Minkowski spacetime

$$ds^2 = -du^2 - 2du\,dr + r^2 \gamma_{AB}\, dx^A dx^B$$

$$\lim_{r \to \infty} g_{uu} = -1$$

$$\lim_{r \to \infty} g_{ur} = -1$$

$$\lim_{r \to \infty} g_{uA} = 0$$

$$\lim_{r \to \infty} g_{AB} = r^2 \gamma_{AB}$$

γ_{AB} : round metric on the sphere.

After a detailed anaylsis the following boundary conditions have been proposed.

$$g_{uu} = -1 + \mathcal{O}(r^{-1})$$

$$g_{ur} = -1 + \mathcal{O}(r^{-2})$$

$$g_{uA} = \mathcal{O}(r^{0})$$

$$g_{AB} = r^2 \gamma_{AB} + \mathcal{O}(r) .$$

The boundary conditions need to ensure good definition of charges. We cannot be too restrictive. We cannot be too general.

metric

$$ds^2 = e^{2\beta} \frac{V}{r} du^2$$

$$- 2 e^{2\beta} du\, dr$$

$$+ g_{AB} \left(dx^A - u^A du\right)\left(dx^B - u^B du\right)$$

where g_{AB} is the metric on the sphere

$$g_{AB} = r^2 \left(\gamma_{AB} + \frac{C_{AB}}{r} + \frac{D_{AB}}{r^2} + \frac{E_{AB}}{r^3} + \cdots \right)$$

since the $\sqrt{\det g_{AB}} = r^2$ there are trace conditions on C_{AB}, D_{AB}, E_{AB}, ...

$$\boxed{\gamma^{AB} C_{AB} = 0}$$

C_{AB} is symmetric traceless tensor which is an arbitrary f^n of the retarded time u and angles.

$C_{AB}(u, x^A)$: $C_{\theta\theta}(u, x^A)$ & $C_{\theta\phi}(u, x^A)$

It is the radiative field. It contains two polarizations

It contains all the information about gravitational radiation at $\mathcal{J}^+$.

Bondi News

$$N_{AB} = \partial_u C_{AB}$$

whatever dynamics is happening in the spacetime, C_{AB} is the imprint of it. It brings the spacetime news at null infinity — at far distances. Hence the name for $\partial_u C_{AB}$ News tensor or Bondi News or Bondi News tensor.

EOM analysis

$$e^{2\beta} = 1 + O\left(\frac{1}{r^2}\right)$$

$$V = -r + 2M + O\left(\frac{1}{r}\right)$$

$$\boxed{e^{2\beta}\frac{V}{r} = -1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)}$$

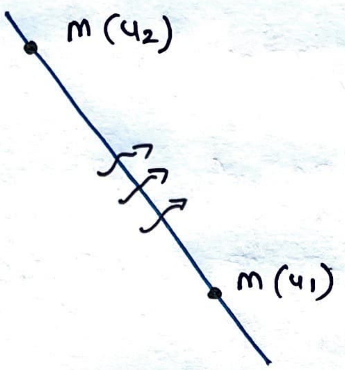
$m(u, x^A)$: Bondi mass aspect

It gives the angular density of energy of the spacetime as measured from a point at $\mathcal{J}^+ (u, x^A)$.

Bondi mass $\int_{S^2_\infty} d^2\Omega \; m(u, x^A) = M(u)$.

$\boxed{\partial_u m(u) \leq 0}$ one can show.

Physically radiation carried by gravitional waves that escapes through $\mathcal{I}^+$ lowers the energy of spacetime.



$$m(u_2) \leq m(u_1)$$

$$m(u \to -\infty) = m_{ADM}$$

The Bondi mass equates the ADM mass defined at spatial infinity.

$$ds^2 = -du^2 - 2du\,dr + r^2 \gamma_{AB}\, dx^A dx^B$$

$$+ \frac{2m}{r} du^2 + r C_{AB}\, dx^A dx^B + D^B C_{AB}\, du\, dx^A$$

$$+ \text{subleading terms.}$$

We analyse the subleading terms and other things in the next lecture.